STRUCTURAL OPTIMIZATION UNDER CONSTRAINTS

OF WHATEVER KIND

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ABSTRACT

The purpose of this paper is to present a general approach to the problem of the minimum mass design of structures which are required to satisfy constraints of a dynamic or aeroelastic nature.

A finite element idealization is employed for this purpose, and the governing non linear equations are solved by an iterative procedure via Pontryagin principle.

Several typical structures are studied as example problems to illustrate the computational algorithm.

1. STATEMENT OF THE PROBLEM

The overall dimensions of the structure, and its specific nature (beam; plate; rod; shell; truss, etc.), are given. It is required to find structural thicknesses $u_k (k = 1,2,...N)$ of the elements E into which the structure is ideally subdivided.

Let $\chi_k(u_k)$ be the weight of E , and denote by χ the vector of the χ_k's, by U the vector of the u_k's. It is required to find U in such a way that the total weight:

$$\Pi = \sum \chi \tag{1}$$

(where $\sum$ is the summing vector [1,1,...1]) be a minimum. In addition, the state vector equations:

$$A_r(U)W_r = 0 \; ; \qquad r = 1,2,...R \tag{2}$$

must be satisfied. In Eq. (2) W_r is the r-th state vector, and the A_r are matrices depending on the "control" U and on some specified or unspecified parameter $\omega_1, \omega_2 ...\omega_s$.

The matrices A_r correspond to the particular behaviour that is wanted to be reproduced in the structure. As seen, therefore, the paper is concerned with linear behaviour, although no real further serious difficulties would arise in considering non linear behaviour. Furthermore, we confine ourselves to examine homogeneous problems, which include however some of the most significant structural features, such as, f.i., free vibrations, Euler's load, aeroelasticity, and so on.

In addition, in problems of real engineering controls have upper and lower limitations. If we stipulate, for two vectors a,b, that $a \geq b$ means that no component of b can exceed the corresponding component of a , we have two given vectors U', U", such that:

$$U' \leq U \leq U" \tag{3}$$

since, in general, different limitations may be valid for different elements.

2. OPTIMALITY EQUATIONS

For optimality, Eqs. (1), (2) are equivalent to find the minimum of the functional:

$$\Pi_a = \sum \chi + \sum_{1}^{R}{}_r \lambda_r^T A_r W_r \tag{4}$$

where each of the Lagrange multiplier vector λ_r must be consistent with the vector equation:

$$A_r^T \lambda_r = 0 \; ; \quad r = 1,...R \tag{5}$$

Obviously, since (2) implies that A_r must be singular, Eq. (5) provides nontrivial values for λ_r . Eq. (4) can be written:

$$\Pi_a = \sum H \tag{6}$$

where H is the Hamiltonian vector, whose k-th component is given by

$$H_k = \chi_k + \sum_{1}^{R}{}_r \lambda_{r,k} (\sum_{1}^{N}{}_h A_{r,kh} W_{r,h}) ; \; k = 1,...N \tag{7}$$

It is known [Bibl.1] that minimizing of Π_a is equivalent to minimizing of every H_k. In other words, among all states and all Lagrange multipliers satisfying (2) (5), the optimal solution is the one for which every H_k is a minimum. This is particularly important in view of bounded controls.

3. CONSERVATIVE FORCES

For many important problems, the structure is subjected to conservative forces only. This happens, f.i., in free vibrations analysis where we have two definite positive quadratic forms for each mode of vibration:

$$T_r = \frac{1}{2} W_r^T M W_r \quad \text{and} \quad \Omega_r = \frac{\omega_r^2}{2} W_r^T G W_r$$

where both mass-matrix M and stiffness matrix G are symmetric. If we prescribe some frequencies $\omega_1, \omega_2, ...\omega_R$, so that R = S, Eqs. (2) read now

$$[G - \omega_r^2 M] W_r = 0 \; ; \quad r = 1,...R \tag{8}$$

so that $A^T = A$, and comparison of Eqs. (2) (5) yields:

$$\lambda_r = W_r \; ; \qquad r = 1,...R \tag{9}$$

an unessential multiplying factor excepted. It is obvious that the property (9) is characteristic of all linear processes, whether they be free vibrations, or not. Therefore, the Hamiltonian vector H reads now:

$$H_k = \chi_k + \sum_1^R{}_r \sum_1^N{}_h A_{r,hk} W_{r,h} W_{r,k} \tag{10}$$

4. ILLUSTRATIVE EXAMPLES

In several problems of aerospace structural engineering one has to deal with "sparse" structures, that are subjected to in-field constraints and to boundary conditions. Thus, the problem of solving such structures is essentially a boundary value problem, that can be transformed into an initial value problem, if the values of a comparatively small number of initial constants are taken as unknown, and adjusted until the final conditions are met.

In structural optimization we may adopt a similar criterion. We start from one side of the structure, with the values which are prescribed, and with an additional number of constraints which allow to transform the problem into an initial value one. At every step, or at every gridpoint, we take a decision, in such a way that H_k (7) is a minimum in the allowable range of the control: it is very important to remark however that such decisions are to be taken solely on the basis of already computed values.

We arrive at the other end of the structure. If the final conditions are not satisfied, we try to change the initial guess values.Several numerical techniques have been developed for this purpose, the most popular of which is the so-called "shooting technique" [Bibl. 2].

Let us now consider in detail the following illustrative problem (Fig. 1): the torsion of a rod, consisting of N elements of rigidity $B_k (k = 0,1...N-1)$* separated by flywheels of inertia J_k; for each of them the equilibrium equation, for the generic mode of frequency ω, reads:

$$\psi_k = -\omega^2 J_k\theta_k + B_{k-1}\frac{\theta_k - \theta_{k-1}}{L_{k-1}} - B_k \frac{\theta_{k+1} - \theta_k}{L_k} = 0 \tag{11}$$

$$k = 1, \dots N-1$$

with the boundary conditions:

$$\psi_o = -\omega^2 J_o\theta_o - B_o \frac{\theta_1 - \theta_o}{L_o} + \beta_o \theta_o = 0$$

$$\psi_N = -\omega^2 J_N\theta_N + B_{N-1} \frac{\theta_N - \theta_{N-1}}{L_{N-1}} + \beta_N \theta_N = 0 \tag{12}$$

(*) Here we number elements from 0 to N-1, and not from 1 to N as said in Art.-1, since we may let the step approach zero, and so the zero-th element will correspond to $x = 0$.

where β_o, β_N, are the end-constraint rigidities

We assume that only one frequency is given, so that Eqs. (11) (12) are the only constraint. The optimality functional:

$$\Pi_a = \sum_o^{N-1}{}_k \chi_k + \sum_o^{N-1}{}_k \psi_k \theta_k \tag{13}$$

can be written by using Eqs. (11) (12) and rearranging them in such a way to obtain the Hamiltonian vector, whose components are now:

$$H_k = u_k L_k - \omega^2 J_k \theta_k^2 + B_k L_k \left(\frac{\theta_{k+1} - \theta_k}{L_k}\right)^2 +$$

$$+ \delta_{ko}\beta_o\theta_o^2 + \delta_{kN}\beta_N\theta_N^2 \tag{14}$$

where

δ = Kronecker's delta ; $k = 0,...N$

We assume for the sake of giving a simple example, that B_k and J_k depend solely on u_k in some specified way.

We start from the first of Eq. (12), which involves θ_o, θ_1. We first remark that we cannot prescribe both of them arbitrarily since, as the state vector must be consistent with the state equation, this would mean a nonoptimal choice of u_o, based on Eq. (12). We must write Eq. (12) as:

$$(\beta_o - \omega^2 J_o)\theta_o - T_o = 0 \tag{15}$$

and provide another arbitrary relationship between θ_o, T_o; as an example, for $\beta_o = \infty$, that implies $\theta_o = 0$, we can select arbitrarily T_o. With these values we enter Eq. (14) for $k = 0$

$$H_o = u_o L_o + (\beta_o - \omega^2 J_o)\theta_o^2 + \frac{L_o}{B_o} T_o^2 \tag{16}$$

and determined the optimum for u_o.

We may now pass to the in-field process, defined by the recursion process:

$$T_k = T_{k-1} - \omega^2 J_k \theta_k$$

$$u_k L_k - \omega^2 J_k \theta_k^2 + \frac{L_k}{B_k} T_k^2 = \min \tag{17}$$

$$\theta_{k+1} = \theta_k + \frac{L_k}{B_k} T_k \quad .$$

We determined successively u_1, T_1, θ_2; u_2, T_2, θ_3, ... u_{N-1}, T_{N-1}, θ_N. If this last set of values is not consistent with the second of Eqs. (12) we must change the value of the arbitrary constant given at the beginning until it is satisfied. As said, this can be obtained through a shooting technique, whose numerical principle is essentially similar to Newton-Raphson procedure and needs not to be recalled here.

For a continuous structure, we have $L_k \to dx$; $J_k = j(x)dx$, etc. The Hamiltonian vector becomes the Hamiltonian function

$$H(x) = u(x) - \omega^2 j\, \theta^2 + \frac{T^2}{B} \quad (18)$$

where

$$T = B\, \frac{d\theta}{dx} \quad (19)$$

The process is essentially the same as described by Eqs. (17), where however the first of them should be replaced by more sophisticated integration formulas, for a better truncation error. We note however that combining Eq. (19) into (18) is illegal, for this would mean that we select a relationship between θ and $\theta + (d\theta/dx)dx$, equivalent, as already noted, to a nonoptimal choice for u_o .

As a second example, we consider the case of Fig. 2, which is essentially the same of Fig. 1, with the only difference that now the structural elements are beams, of rigidity EI_k . We denote by w_k the deflection of the mass M_k , and by ϕ_k its rotation. The equations of equilibrium are written:

$$F_k - \frac{L_k}{2} V_k = - F_{k-1} + \frac{L_{k-1}}{2} V_{k-1}$$

$$\frac{3F_k}{L_k} - V_k = \omega^2 M_k w_k - V_{k-1} + \frac{3F_{k-1}}{L_{k-1}} + \frac{6EI_k}{L_k^2}\phi_k - \frac{6EI_{k-1}}{L_{k-1}^2}\phi_{k-1} \quad (20)$$

where:

$$V_k = \frac{12EI_k}{L_k^3}(w_{k+1} - w_k) \; ; \; F_k = \frac{4EI_k}{L_k}(\phi_k + \frac{\phi_{k+1}}{2}) \quad (21)$$

We assume, for sake of giving a simple example, to have to do with a simply supported beam, so the boundary conditions to Eqs. (20)are:

$$w_o = 0 \; ; \quad \phi_o = - \frac{\phi_1}{2} \quad (22)$$

$$w_N = 0 \; ; \quad \phi_N = - \frac{\phi_{N-1}}{2} \quad (23)$$

The Hamiltonian vector is now written:

$$H_k = u_k L_k + \frac{12EI_k}{L_k^3}[w_{k+1} - w_k]^2 + \frac{4EI_k}{L_k}[\phi_k^2 + \phi_{k+1}^2 + \phi_k\phi_{k+1}] + \frac{12EI_k}{L_k^2}(w_k - w_{k+1})(\phi_k + \phi_{k+1}) - \omega^2 M_k w_k^2 \quad (24)$$

and, by introducing the notations (21), it can be written:

$$H_k = u_k L_k + \frac{L_k^3}{12EI_k}V_k^2 + \frac{12EI_k}{L_k}\phi_k^2 + \frac{L_k F_k^2}{EI_k} - L_k^2\frac{V_k F_k}{2EI_k} - \omega^2 M_k w_k^2 + \phi_k(L_k V_k - 2F_k) \quad (25)$$

having deleted terms which do not have influence on the optimal choice. Here again we must start by giving arbitrary values to V_o , ϕ_o and determining u_o from (24),H_o=min. Then we are ready for the infield process, defined by Eqs. (20) (25) (21) in the order. If w_N, ϕ_N are not consistent with (23) we have to change V_o, ϕ_o, until such conditions are fullfilled.

Again, the following considerations may be applied to the continuous structure defined by:

$$\frac{d^2F}{dx^2} = \omega^2 \mu w$$

$$H = u + \frac{F^2}{EI} - \omega^2 \mu w^2 \quad (26)$$

$$F = E\, I\, \frac{d^2 w}{dx^2}$$

and the verification that, as $L_k \to 0$, Eqs. (20) (21) (25) approach Eqs. (26) is left to the reader. Again, of course, in actual computations, more sophisticated processes of numerical integration should be introduced. If also a constant axial compressive load P is acting on the structure, Eqs. (26) may be generalized to yield:

$$\frac{d^2F}{dx^2} = \omega^2 \mu w - Pw''$$

$$H = u + \frac{F^2}{EI} - \omega^2 \mu w^2 + Pw'^2 \quad (27)$$

$$F = EI\, \frac{d^2 w}{dx^2}$$

5. NUMERICAL RESULTS

The above described procedure was applied to some specific numerical cases[(1)]. Fig. 3 refers to the optimum thickness distribution for torsional vibrations (or aeroelastic divergence) of a rod with $B = U, j = 1$ (Eq. 18), when $\omega = \pi/2$ is the given eigenvalue.

No upper bound is prescribed ($U'' = \infty$); the computed optimum thickness refers to several different lower bounds U' . The horizontal straight line labelled $U = 1$ corresponds to the nonoptimal constant thickness distribution. Each distribution has an initial curved arc, followed by a constant arc; transition points, for every value of U' are described by the line AC.

(1) The results of Fig. 3 three 9 are taken from Ref. 3 the presentation of them is due to the courtesy of "L'Aerotecnica, Missili, Spazio". In this number, U=u.

For the same case, further results are given in Fig. 4. Here we have a variation of maximum thicknesses, (max U), transition point (x*), total weight (Π) vs. U'.

The Π curve is very interesting. It can be seen that the maximum weight economy is obtained as one allows minimum thickness to go to zero, and such a maximum economy is of 18%.

Fig. 5 refers to the same problem of Fig. 3, 4 when an upper bound (U" = 1) is prescribed. The optimum thickness of Fig. 5 has now two transition points, x* and x**, whose values are shown in Fig. 6, together with the total weight Π , again compared with the constant, nonoptimal, value Π = 1. The weight economy is now less than in the unbounded case; thus, imposing an upper limit reduces structural ligthness, but the results are doubtlessly more meaningful from a practical point of view.

The results of Fig. 7, 8 refer to another case of vibrating where now the thickness U enters also in the mass distribution. For the constant thickness the eigenvalue under concern would be $\omega = \pi\sqrt{2}/4$, for which the optimum distribution of Fig. 7 have been calculated for the various minimal thickness, while the upper bound has been kept constant and equal to unity.

For sake of comparison, also some more general cases were treated. Fig. 9 summarizes the results for a vibrating rod with $B = A(x)U$; $j = \beta(x)+U\alpha(x)$, for some expressions of the functions A, β, α.

On the right hand vertical straight line, it is possible to read the values of the total weight of the cases considered in the example.

We shall now consider the case of a fourth order equation. Fig. 10 refers to Euler's column which actually can be treated as a second-order equation. The analysis was done for the sake of comparison and check of the method on multi-constant approach. Fig. 11 gives for the same structure maximum thickness and optimal weight: obviously, as U' approaches unity, both curves shall approach unity too. Fig. 12 provides similar results for constant fixed lower bound (U' = 0.1) and with several upper limits.

The group of Figs. 13, 14, 15 refer to an actual fourth-order equation. Here we have to do with a simply supported vibrating flexural beam with $EI = U^3$; $P = 0$; $\mu = U$; $\omega = \pi$, no upper bound, several lower bounds. As is seen, Fig. 13, the thickness distribution, is almost entirely nonsensitive to the minimal thickness imposed; except, of course, the regions where the thickness would be less than the prescribed value. This impression is substantiated by Fig. 14, where it can be seen how small a gain in weight can be obtained by reducing minimal thickness.

Finally, Fig. 15 refers to the same structure of Figs. 13, 14 where now we impose a fixed lower bound (U' =.1) and several upper bounds. The distribution is seen now to be much more sensitive to U".

6. NON CONSERVATIVE FORCES

For non-conservative forces, the matrix A is not symmetric, and the property (9) does not hold. Therefore, we must shoot a double number of initial constants.

We refer, in particular, to aeroelasticity, a typical problem of which is to determine an optimum weight structure when critical dynamic pressure is given. Here we have:

$$[G - \omega^2 M + \sigma_c L]W = 0 \tag{28}$$

where σ_c is the dynamic pressure parameter, and L is the aerodynamic matrix. The complete set of equations defining the problem is now, in addition to Eq. (28):

$$[G - \omega^2 M + \sigma_c L^T]\lambda = 0 \tag{29}$$

and

$$u + \lambda^T [G - \omega^2 M + \sigma_c L]W = \text{opt.} \tag{30}$$

$$\Delta = \mathrm{Det}[G - \omega^2 M + \sigma_c L] = 0; \qquad \left(\frac{d\Delta}{d\omega^2}\right) = 0 \tag{31}$$

It is seen therefore that now also the parameter ω is unknown, and there is the further condition (31).

The shooting technique may be employed also in this case. To visualize this, let us consider again the structure of Fig. 2, where now also an aeroelastic piston-theory pressure is acting. The equation valid in the generic k-th element of the structure between masses M_k and M_{k+1} is now:

$$\frac{d^4 w}{dx^4} + \nu_k^3 \frac{dw}{dx} = 0 \ ; \quad (x_k \le x \le x_{k+1}) \tag{32}$$

where $\nu_k^3 = \frac{\sigma}{EI_k}$; and for the Lagrange multiplier $\lambda(x)$, we have the equation:

$$\frac{d^4 \lambda}{dx^4} - \nu_k^3 \frac{d\lambda}{dx} = 0 \ ; \quad (x_k \le x \le x_{k+1}) \tag{33}$$

By solving Eq. (32) with the values of w and dw/dx at the ends of the element, we can write the equation of equilibrium for the k-th mass under the form:

$$w_{k+1}\eta_k^{(10)} + w_k[\eta_k^{(00)} - \eta_{k-1}^{(11)}] - w_{k-1}\eta_{k-1}^{(01)} + \phi_{k+1}\theta_k^{(10)} + \phi_k[\theta_k^{(00)} - \theta_{k-1}^{(11)}] - \phi_{k-1}\theta_{k-1}^{(01)} = 0$$

$$w_{k+1}\varepsilon_k^{(10)} + w_k[\varepsilon_k^{(00)} - \varepsilon_{k-1}^{(11)}] - w_{k-1}\varepsilon_{k-1}^{(01)} + \phi_{k+1}\zeta_k^{(10)} + \phi_k[\zeta_k^{(00)} - \zeta_{k-1}^{(11)}] - \phi_{k-1}\zeta_{k-1}^{(01)} = \omega^2 M_k w_k$$

(34)

The coefficients are given in the Appendix. Suffice here to say they are functions of ν_k.

A similar equation holds for the Lagrange multiplier μ_k of the first of Eqs. (34) and for the Lagrange multiplier λ_k of the second of Eqs. (34), where however the coefficients are calculated for $-\nu_k$ instead of ν_k. The Hamiltonian reads now:

$$H_k = u_k - \omega^2 M_k w_k^2 + \mu_k [w_{k+1}\eta_k^{(10)} + w_k\eta_k^{(00)} + \phi_{k+1}\theta_k^{(10)} + \phi_k\theta_k^{(00)}]$$

$$- \mu_{k+1}[w_{k+1}\eta_k^{(11)} + w_k\eta_k^{(01)} + \phi_{k+1}\theta_k^{(11)} + \phi_k\theta_k^{(01)}] +$$

$$+ \lambda_k[w_{k+1}\varepsilon_k^{(10)} + w_k\varepsilon_k^{(00)} + \phi_{k+1}\zeta_k^{(10)} + \phi_k\zeta_k^{(00)}] -$$

$$- \lambda_{k+1}[w_{k+1}\varepsilon_k^{(11)} + w_k\varepsilon_k^{(01)} + \phi_{k+1}\zeta_k^{(11)} + \phi_k\zeta_k^{(01)}] . \tag{35}$$

We solve Eqs. (34) for w_{k+1} , ϕ_{k+1} in terms of updated state quantities, and coefficients depending on I_k. Similarly we solve the corresponding equation for λ_{k+1}, μ_{k+1} ; now we can enter Eq. (35) that contains only updated quantities and I_k, u_k, M_k. Here we can take the optimum decision; obviously, in this case, we must "shoot" four initial constants. The procedure is the same as described at n. 4.

It is left to the reader to check that all the expressions reduce to those already given at n. 4 as $\sigma \to 0$.

For a continuous structure, we must solve the differential system:

$$\frac{d^2}{dx^2}\left(EI\frac{d^2w}{dx^2}\right) + \sigma\frac{dw}{dx} - \omega^2\mu w = 0$$
$$\frac{d^2}{dx^2}\left(EI\frac{d^2\lambda}{dx^2}\right) - \sigma\frac{d\lambda}{dx} - \omega^2\mu\lambda = 0 \tag{36}$$

with the pertinent initial and final, boundary and transversality condition. An excellent paper by Weishaar [4] deals with this problem for a simply supported sandwich panel; although not proved, it is very likely that the optimum thickness should be symmetrical, although the mode is not.

Figs. 16, 17, give the results of the calculations on the same problem by the present method. It is seen that there is a rather large discrepancy, due probably to some instability effects of the integration method used by us. The correction of the program is now under completion, to adapt it also to two-dimensional cases.

7. MULTIDIMENSIONAL STRUCTURES: FINITE ELEMENTS TECHNIQUE.

To apply the shooting technique to multidimensional structures, one has to establish a "starting front" and walk in a direction transverse to such front until one reaches the final front where the boundary conditions are to be checked. This can be rather difficult for generic shapes of boundary; therefore, a careful selection of the elements and of gridpoints can help very much in a substantially improved technique. No general rules can be given in this context, but simply one has to rely upon the ingenuity and the ability of the analyst.

In any case, transversal boundary conditions may change a little the technique as described in the previous Articles. Let us consider, for example, the structure of Fig. 18, i.e., a rectangular plate, formed by rectangular panels, each of constant thickness, with vibrating masses at the gridpoints. We want to generalize, to the two-dimensional case, the one-dimensional case of Art. 4, i.e., the minimum weight plate of a given frequency.

We firstly note, Fig. 19, that finite element technique provides for each panel a matrix depending on the panel thickness, which relates corner forces with corner displacements. Furthermore, the components of the Hamiltonian vector relative to the panel under concern will depend on the panel thickness and on the set of (Fig. 19) the displacements at the panel corners. However, we cannot eliminate the quantities at the (k+1)th row, as done in the examples of Art. 4, since the equations of equilibrium of the single panels and masses are not independent, and, furthermore, also the lateral boundary conditions must be considered. In this case, we have to use a successive approximations method. We give a first try or guess approximation for the various thicknesses of the k-th row of panels Fig. 19; then, since all the displacements at the k-th row are known, a very simple linear problem allows to go one row ahead, since boundary conditions and A and B are prescribed. Now we can enter the optimality equations, and correct for the initial guess thickness - distribution, and so on. For a continuous plate, the Hamiltonian reads:

$$H(x,y) = u(x,y) - \omega^2\mu[u(x,y)]w^2 + \\ + \frac{F_x^2 + F_y^2 - 2F_xF_y\nu + 2(1-\nu)F_{xy}^2}{D[u(x,y)]} \tag{37}$$

Here we have to choose, as shooting quantities the values of $\partial w/\partial x$, $\partial^3 w/\partial x^3$ (if, f.i. , the plates are simply supported at $x = 0$, so there we know $w = 0$; $\partial^2 w/\partial x^2 = 0$) , which are functions of y only, and walk in the x-direction (Fig. 20). At every step Δx (Fig. 21), we have to solve the problem of minimizing (37) for all F_x, F_y, which can be obtained by solving the variable-thickness plate equation when four quantities are given on the sides PQ, and two quantitites each at the sides MP, NQ. Check of boundary condition, and updating of the shooted quantities is performed at the $x = a$ edge of Fig. 20.

It is obvious, however, that the optimum structure will come out of a judicious choice of the elements and of the adequate representation.

Furthermore, it is clear that constraints are more important than optimality. In other words, if an "optimum" structure is difficult, one can content himself with a "very good" one, provided that, however, structural consistency is assured. For this purpose, it is very important to have an efficient method of calculating frequencies of a given structure. Our Institute has gained considerable experience in this area, by applying very sophisticated programs (such as f.i., NASTRAN) to very sophisticated structure. A typical example, the Spoot Beam Antenna, built by the Italian society "SELENIA"; our Institute has performed the structural analysis, by using the model of Fig. 22 and obtaining results in excellent agreement with experimental results. The author's opinion is that this is a very efficient tool as an intermediate, often repeated step in structural optimization.

APPENDIX

Consider Eq. (32) valid in the range $0 \le x \le L$, with the end conditions:

$$x = 0; \quad w = w_o, \quad \frac{dw}{dx} = \phi_o \qquad\qquad x = L \quad w = w_1, \quad \frac{dw}{dx} = \phi_1 \tag{1}$$

Letting $\nu^3 = \frac{\sigma L^3}{EI}$, we may write:

$$w(\xi) = w_o Q_o(\xi) + w_1 Q_1(\xi) + \frac{\phi_o}{\nu} P_o(\xi) + \frac{\phi_1}{\nu} P_1(\xi) \tag{2}$$

where $\xi = \frac{x}{L}$, and

$$Q_o(\xi) = 1 + \frac{\phi_c(\xi)\dot\phi_b(1) - \phi_b(\xi)\dot\phi_c(1)}{\Delta} \; ; \; Q_1(\xi) = 1 - Q_o(\xi)$$

$$P_o(\xi) = \psi(\xi) + \dot\psi(1)\frac{\phi_b(\xi)\phi_c(1) - \phi_c(\xi)\phi_b(1)}{\Delta} + $$

$$+ \psi(1)\frac{\phi_c(\xi)\dot\phi_b(1) - \phi_b(\xi)\dot\phi_c(1)}{\Delta}$$

$$P_1(\xi) = \frac{\phi_b(1)\phi_c(\xi) - \phi_c(1)\phi_b(\xi)}{\Delta}\nu \tag{3}$$

$$\Phi_B(\xi) = e^{\frac{\nu\xi}{2}} \text{sen}\left(\frac{\sqrt{3}\nu\xi}{2}\right) - \frac{\sqrt{3}}{2}\psi(\xi); \; \psi(\xi) = 1 - e^{-\nu\xi}$$

$$\phi_c(\xi) = -\frac{1}{2}\psi(\xi) + e^{\frac{\nu\xi}{2}} \cos\left(\frac{\nu\xi\sqrt{3}}{2}\right) - 1; \; (\;)^{\cdot} = \frac{d(\;)}{d\xi}$$

$$\Delta = \phi_b(1)\dot\phi_c(1) - \phi_c(1)\dot\phi_b(1)$$

From Eq. (2) we can calculate shears and bending moments at the end of the bar. Letting:

$$\varepsilon^{(ij)} = \frac{EI}{L^3}\left(\frac{d^3Q_i}{d\xi^3}\right)_{\xi=\xi_j} ; \; \zeta^{(ij)} = \frac{EI}{\nu L^3}\left(\frac{d^3P_i}{d\xi^3}\right)_{\xi=\xi_j}$$

$$\eta^{(ij)} = \frac{EI}{L^3}\left(\frac{d^2P_i}{d\xi^2}\right)_{\xi=\xi_j} ; \; \theta^{(ij)} = \frac{EI}{\nu L^3}\left(\frac{d^2P_i}{d\xi^2}\right)_{\xi=\xi_j} \tag{5}$$

$$(i,j=0,1)$$

we have:

$$V_o = w_o\varepsilon^{(00)} + w_1\varepsilon^{(10)} + \phi_o\zeta^{(00)} + \phi_1\zeta^{(10)}$$
$$V_1 = w_o\varepsilon^{(01)} + w_1\varepsilon^{(11)} + \phi_o\zeta^{(01)} + \phi_1\zeta^{(11)} \tag{6}$$

$$F_o = w_o\eta^{(00)} + w_1\eta^{(10)} + \phi_o\theta^{(00)} + \phi_1\theta^{(10)}$$
$$F_1 = w_o\eta^{(01)} + w_1\eta^{(11)} + \phi_o\theta^{(01)} + \phi_1\theta^{(11)} \tag{7}$$

The same relationships hold, of course, also for the generic k-th span of the structure of Fig. 3, when $w_o = w_k$; $w_1 = w_{k+1}$; $\phi_o = \phi_k$; $\phi_1 = \phi_{k+1}$, and the proper value of $\nu_k^3 = \sigma L_k^3/EI_k$ has been introduced.

SYMBOLS

- j → rotary inertia per unit length
- u → thickness
- w → flexural displacement
- x,y coordinates of plate or beam or rod
- A → structural matrix
- B → torsional stiffness
- D → flexural stiffness of plate
- E → modulus of elasticity
- *E* → element
- F → bending moment
- G → stiffness matrix
- H → Hamiltonian
- I → moment of inertia of beam section
- L → element length
- *L* → aeroelastic matrix
- M → mass matrix, or single mass
- N → number of elements
- P → compressive load
- R → number of constraints
- S → number of parameters
- T → torque
- *T* → kinetic energy
- U → thickness vector
- V → Eq. (21)
- W → state vector
- β → end constraint rigidity
- ε,ζ,η,θ → coefficients (see Appendix)
- θ → angle of rotation (Art. 4)
- λ → Lagrange multiplier
- μ → Lagrange multiplier or mass per unit of length
- ν → $\sqrt[3]{\sigma/EI}$, or Poisson's ratio
- σ → dynamic pressure parameter
- φ → flexural rotation
- χ → weigth of element
- ψ → Eq. (11)
- ω → given or ungiven parameter

$\Delta \rightarrow$ Eq. (3)
$\Pi \rightarrow$ total weight
$\sum \rightarrow$ summing vector
$\Omega \rightarrow$ potential energy

Subscripts

h,k,j $\rightarrow$ number of order of the element
r $\rightarrow$ number of order of constraints
a $\rightarrow$ augmented
x,y $\rightarrow$ in the x or y direction

Superscripts

' = minimum
" = maximum
T = transfer of a vector or matrix

ACKNOWLEDGEMENTS

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D I S C U S S I O N

J. Solvey (Aeronautical Research Laboratories, Melbourne, Australia): If this procedure refers to mono-tonically changing parameters, how do you treat the change in case of step-function?

P. Santini and R. Barboni: In effect the method deals with finite changes of the parameter.

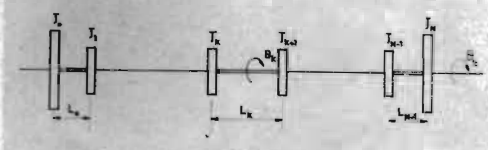

FIG. 1

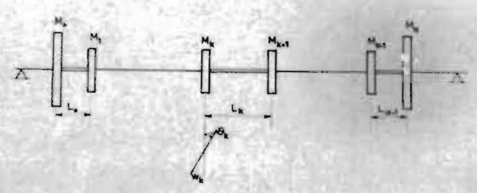

FIG. 2

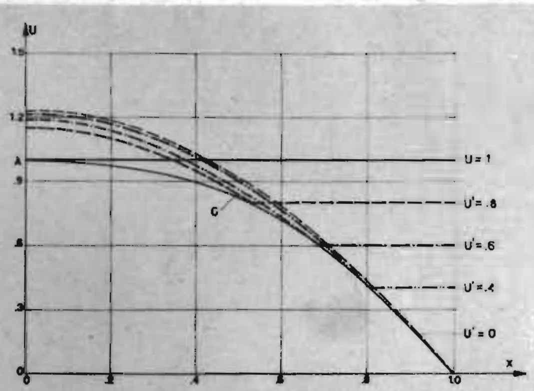

FIG. 3 OPTIMUM THICKNESS DISTRIBUTION OF FIXED-FREE VIBRATING ROD. $B = U$; $j = 1$; $\omega = \frac{\pi}{2}$; $U'' = \infty$ _

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

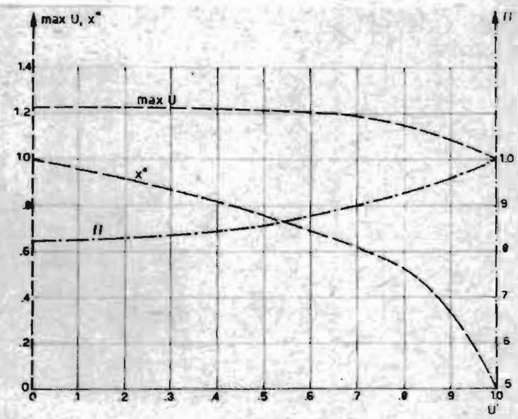

FIG. 4 OPTIMUM TOTAL WEIGHT (Π) FOR THE STRUCTURE OF FIG. 3. Π=1 IS THE NON OPTIMAL CONSTANT THICKNESS. max U, = MAXIMUM THICKNESS ; x* = TRANSITION POINT _

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

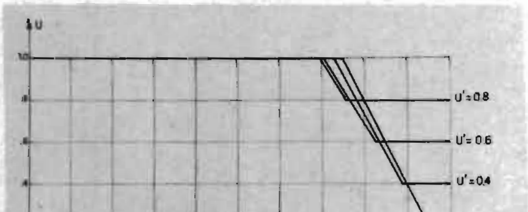

FIG. 5 OPTIMUM THICKNESS DISTRIBUTION OF FIXED-FREE VIBRATING ROD_ $B = U$; $j = 1$; $\omega = \frac{\pi}{2}$; $U'' = 1$ _

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

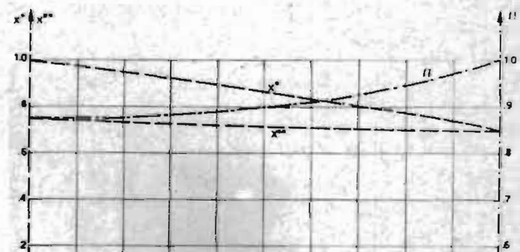

FIG. 6 OPTIMUM TOTAL WEIGHT (Π) AND TRANSITION POINTS FOR THE STRUCTURE OF FIG. 5.

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

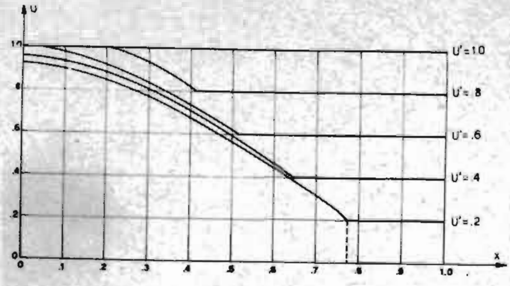

FIG 7 OPTIMUM THICKNESS DISTRIBUTION OF FIXED-FREE VIBRATING ROD
(B = U ; j = 1 + U ; $\omega = \frac{\pi\sqrt{2}}{4}$; U"=1)

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

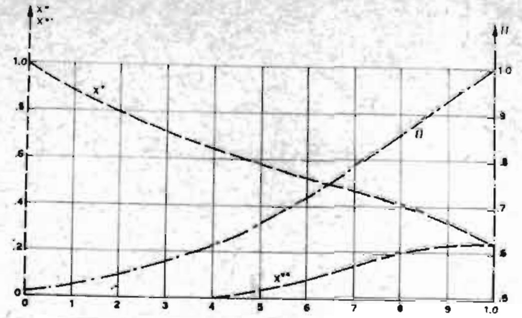

FIG. 8 OPTIMUM TOTAL WEIGHT (Π) AND TRANSITION POINTS FOR THE STRUCTURE OF FIG.7_

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

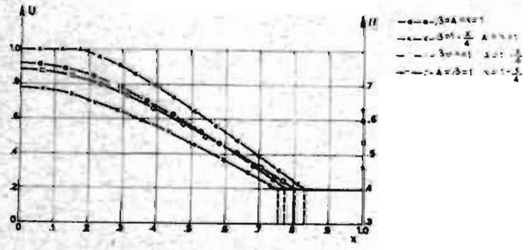

FIG. 9 OPTIMUM THICKNESS DISTRIBUTION OF FIXED-FREE VIBRATING ROD_
B = AU ; j = β + α U ; $\omega = \frac{\pi\sqrt{2}}{4}$; U'= 0.4 ; U"= 1.

courtesy"L'AEROTECNICA MISSILI E SPAZIO"

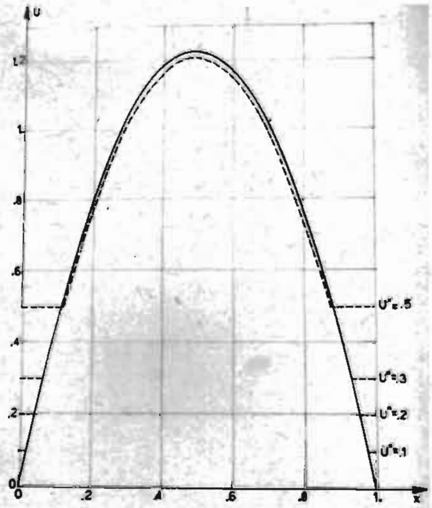

FIG. 10 OPTIMUM THICKNESS DISTRIBUTION FOR SIMPLY SUPPORTED EULER'S COLUMN -
[EI = U ; μ = 0 ; P = π^2 ; Eqs.(27); U" = ∞]

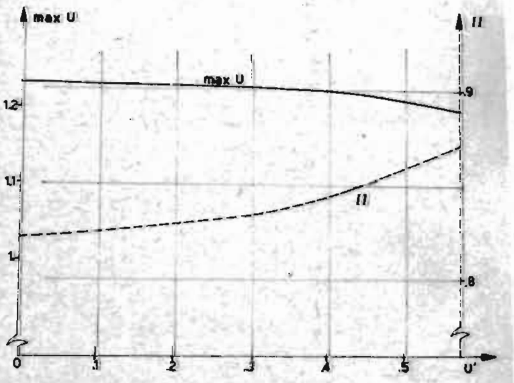

FIG. 11 MAXIMAL THICKNESS (MAX U) AND OPTIMAL WEIGHT FOR THE STRUCTURE OF FIG. 10 _

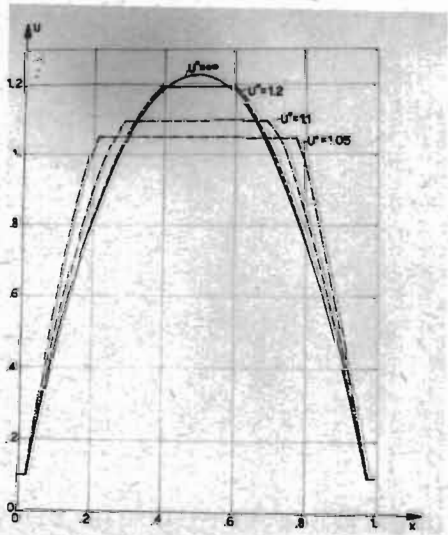

FIG. 12 OPTIMUM THICKNESS DISTRIBUTION FOR SIMPLY SUPPORTED EULER'S COLUMN_ [EI=U ; μ=0 ; P=π^2; Eqs.(27); U'=0.1]

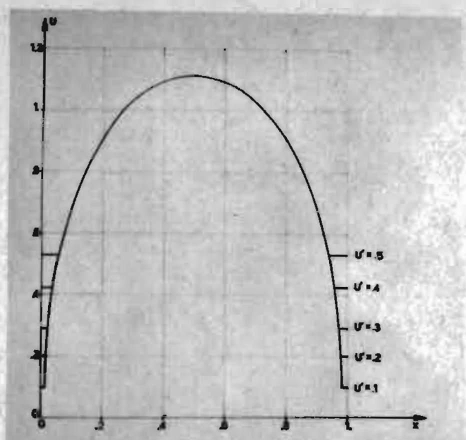

FIG. 13 OPTIMUM THICKNESS DISTRIBUTION FOR VIBRATING FLEXURAL BEAM. [EI = U²; P=0; μ=U; ω=π²; Eqs. (27); U'=var]

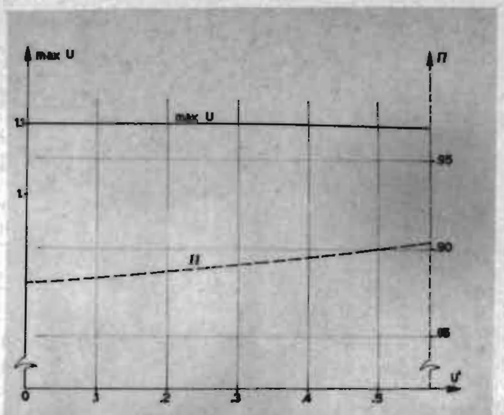

FIG. 14 MAXIMUM THICKNESS (max U) AND OPTIMAL WEIGHT (Π) FOR THE STRUCTURE OF FIG. 13.

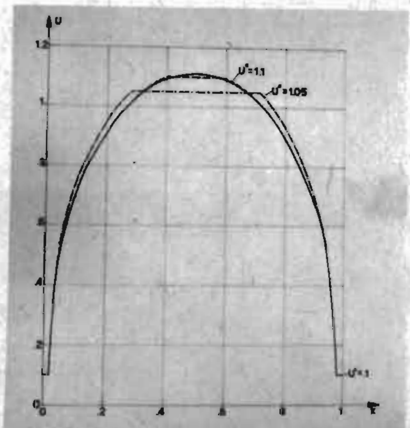

FIG. 15 OPTIMUM THICKNESS OF VIBRATING FLEXURAL BEAM. [EI = U²; P=0; μ=U; ω=π²; Eqs. (27); U'=.1]

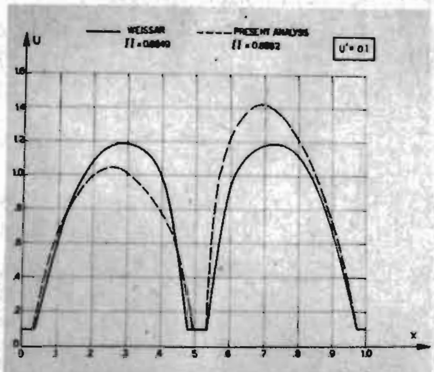

FIG. 16 OPTIMUM THICKNESS DISTRIBUTION FOR FLUTTERING SIMPLY SUPPORTED SANDWICH PANEL.
σ_{cr} = 342.20 ; U' = .1 ; D = U ; μ = 1 + U

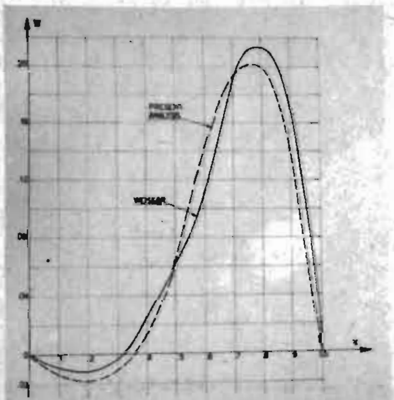

FIG. 17 AEROELASTIC CRITICAL MODE FOR THE STRUCTURE OF FIG. 16.

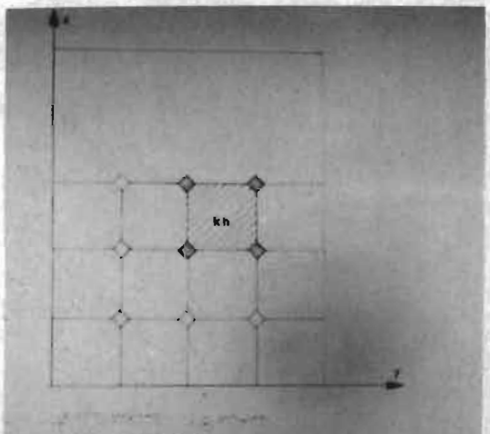

FIG. 18

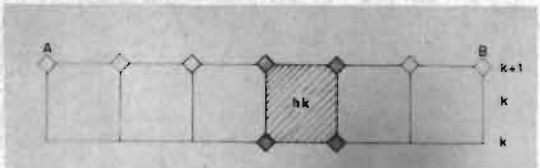

FIG. 19

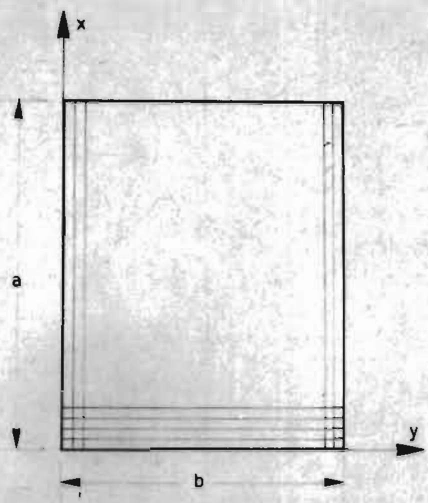

FIG. 20

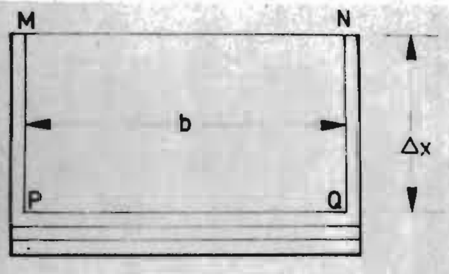

FIG. 21

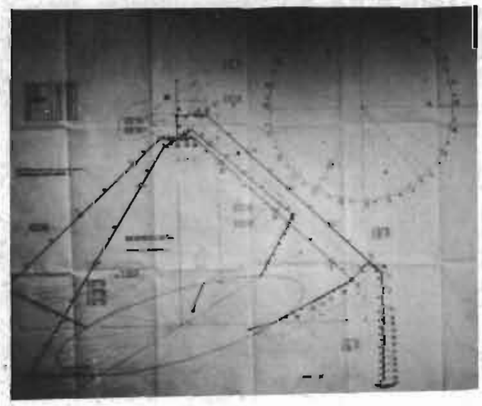

FIG. 22